Stability of cylindrical Couette flow in the presence of an oscillating axial magnetic field

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The linear stability of cylindrical Couette flow of an electrically conducting fluid in the presence of an axial magnetic field is examined, where the magnetic field has a small oscillatory component imposed on a steady value. The effect of the field modulation on the threshold of instability is studied for different values of gap width, Chandrasekhar number, magnetic Prandtl number, and oscillation frequency. Modulation is found to have a stabilizing effect for low values of the Chandrasekhar number, a destabilizing effect for intermediate values, and again a stabilizing effect for still higher values of the Chandrasekhar number. The effect of modulation is found to be almost independent of the magnetic Prandtl number and the modulation frequency.

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I. INTRODUCTION

The onset of instability in modulated Taylor-Couette flow has been studied extensively both experimentally [1–5] and theoretically [6–14]. This was inspired by the classical inverted pendulum, which can be stabilized by oscillation of the point of support, for suitable values of amplitude and frequency of oscillation, which raised the question whether modulation of a suitable parameter can lead to increased stability of Taylor-Couette flow. Experimental studies were carried out in which a steady rotation was provided to the inner cylinder and an oscillating angular velocity was imposed on either the inner [1–4], or the outer [5] cylinder. Since there was a qualitative difference in the initial findings [1–3], a number of theoretical studies were carried out in order to resolve the discrepancy. From these theoretical studies and the later experimental studies [4,5] a consensus seems to have emerged that modulation of the inner cylinder rotation rate is destabilizing while modulation of the outer cylinder rotation rate is stabilizing.

The stability of some other fluid configurations with a time-periodic unperturbed state has also been studied. Theoretical as well as experimental studies have been carried out for the Rayleigh-Bénard problem in which either the temperature or the gravitational force is externally modulated [15–21]. The effect of a steady magnetic field on the Taylor-Couette flow or Rayleigh-Bénard problem has been studied [22]. A few theoretical studies of the effect of an oscillating magnetic field on hydromagnetic stability have also been reported [23,24]. In Ref. [23] the effect of an oscillating magnetic field on the stability of parallel flows was studied, while in Ref. [24] the effect of a modulated magnetic field on convection in a magnetic fluid was studied. To our knowledge no studies on the effect of an oscillating magnetic field on the hydromagnetic Taylor-Couette problem have been reported. In this paper, we report the results of a theoretical study of this configuration carried out using the method of Refs. [6,14]. We hope this study will be interesting since it can provide another configuration with an oscillatory unperturbed state for detailed comparison between theory and experiment, and especially since a modulated magnetic field should be much easier to generate in experiments than a modulation of the rotation rate. The methods for creating a modulation of rotation rate for experimental studies on the modulated Taylor-Couette flow are described in Refs. [2,4]. On the other hand, an oscillating magnetic field can be produced quite easily by passing an oscillating current through a solenoidal coil generating the magnetic field. A similar method of generating a modulated temperature by passing an oscillating current through a heating element has been used in experimental studies of convection with boundaries of oscillating temperature [21,25].

In this study we carry out a theoretical analysis of the stability of an incompressible, viscous, electrically conducting fluid between two coaxial cylinders, the inner rotating with constant angular velocity and the outer held stationary, in the presence of an axial magnetic field which has an oscillating component imposed on a steady value. In Sec. II we give the mathematical formulation for the stability study and in Sec. III we report the results.

II. FORMULATION

We consider an incompressible, viscous, electrically conducting fluid between two coaxial cylinders in the presence of an externally imposed, modulated axial magnetic field. The governing equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla\Pi + \frac{1}{\rho\mu}\mathbf{B}\cdot\nabla\mathbf{B} + \nu\nabla^2\mathbf{v}, \quad (1a)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{v} + \lambda\nabla^2\mathbf{B}, \quad (1b)$$

$$\nabla\cdot\mathbf{v} = 0, \quad (1c)$$

$$\nabla\cdot\mathbf{B} = 0, \quad (1d)$$

where $\mathbf{v}$ is the fluid velocity, $\mathbf{B}$ is the magnetic field, $\Pi = p + B^2/2\mu$ is the total pressure, p is the fluid pressure, μ is the magnetic permeability, and ρ, ν, and λ are the density, kinematic viscosity, and magnetic diffusivity of the fluid.

We choose a cylindrical coordinate system (r,θ,z) with the z axis along the axis of the cylinders. We assume that an azimuthal velocity is imparted to the fluid by rotation of the inner cylinder so that the unperturbed state is given by

$$\mathbf{v} = V(r,t)\mathbf{e}_\theta, \quad \Pi = P(r,t), \quad \mathbf{B} = B_z(r,t)\mathbf{e}_z. \tag{2}$$

Substituting in Eq. (1), we require

$$\frac{V^2}{r} = \frac{1}{\rho}\frac{\partial P}{\partial r}, \tag{3a}$$

$$\frac{\partial V}{\partial t} = \nu\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2}\right), \tag{3b}$$

$$\frac{\partial B_z}{\partial t} = \lambda\left(\frac{\partial^2 B_z}{\partial r^2} + \frac{1}{r}\frac{\partial B_z}{\partial r}\right). \tag{3c}$$

We assume that the inner cylinder, of radius R_1, is rotated with constant angular velocity Ω_1, while the outer cylinder, of radius R_2, is stationary. Then

$$V = \begin{cases} \Omega_1 R_1 & \text{at } r = R_1, \quad (4a)\\ 0 & \text{at } r = R_2. \quad (4b)\end{cases}$$

Since the equations for V and B_z are not coupled we can assume that V is independent of time. Then, solving for V, we obtain

$$V = C_1 r + \frac{C_2}{r}, \tag{5}$$

where

$$C_1 = -\frac{\Omega_1 R_1^2}{R_2^2 - R_1^2}, \quad C_2 = \frac{\Omega_1 R_1^2 R_2^2}{R_2^2 - R_1^2}.$$

If we consider a magnetic field modulation with frequency ω then the term on the left-hand side of Eq. (3c) is $\sim\omega B_z$ while the terms on the RHS are $\sim\lambda B_z/R_1^2$, assuming that R_1 is a typical length scale. The ratio of the term on the LHS to the terms on the RHS is $\sim\omega R_1^2/\lambda = \sigma\,\mathrm{Pm}$, where $\sigma = \omega R_1^2/\nu$ is a nondimensional frequency parameter and $\mathrm{Pm} = \nu/\lambda$ is the magnetic Prandtl number of the fluid. We assume $\sigma\,\mathrm{Pm} \ll 1$. This is reasonable since for liquid metals usually $\mathrm{Pm} \ll 1$ and consequently the condition $\sigma\,\mathrm{Pm} \ll 1$ holds even for $\sigma \sim 1$. In our numerical study we have restricted attention to the range $\sigma \sim 1$, i.e., modulation frequencies of the order of (viscous diffusion time)$^{-1}$. With this assumption the term on the LHS of Eq. (3c) is very small compared to the terms on the RHS and can be neglected. It is then consistent to assume

$$B_z = B_0(1 + \epsilon\cos\omega t), \tag{6}$$

where B_0 is a constant and ϵ is a small parameter which gives the magnitude of the modulation in the magnetic field. We now transform to nondimensional variables using

$$t \to \frac{\tau}{\omega}, \quad (r,z) \to R_1(r,z), \quad V \to \Omega_1 R_1 V, \quad B_z \to B_0 B_z. \tag{7}$$

Then the unperturbed solution is given by

$$V = C_1 r + \frac{C_2}{r}, \quad B_z = 1 + \epsilon\cos\tau, \tag{8}$$

where

$$C_1 = -\frac{\eta^2}{1-\eta^2}, \quad C_2 = \frac{1}{1-\eta^2}$$

and $\eta = R_1/R_2$.

We now consider a small perturbation from this unperturbed state given, in dimensional form, by

$$\mathbf{v} = u\mathbf{e}_r + (V+v)\mathbf{e}_\theta + w\mathbf{e}_z, \quad \Pi = P + p,$$

$$\mathbf{B} = b_r\mathbf{e}_r + b_\theta\mathbf{e}_\theta + (B_z + b_z)\mathbf{e}_z. \tag{9}$$

We again transform the perturbations to nondimensional variables by

$$(u,v,w) \to \frac{\lambda}{R_1}\left(u, \frac{\Omega_1 R_1^2}{\nu}v, w\right), \quad p \to \frac{\rho\nu\lambda}{R_1^2}p,$$

$$(b_r, b_\theta, b_z) \to B_0\left(b_r, \frac{\Omega_1 R_1^2}{\nu}b_\theta, b_z\right). \tag{10}$$

Assuming that the perturbations are axisymmetric, on substituting from Eqs. (9) and (10) in Eq. (1) and linearizing in the perturbations, we obtain, in terms of nondimensional variables,

$$\sigma\frac{\partial u}{\partial\tau} - T\frac{V}{r}v = -\frac{\partial p}{\partial r} + \left(DD_* + \frac{\partial^2}{\partial z^2}\right)u + Q(1+\epsilon\cos\tau)\frac{\partial b_r}{\partial z}, \tag{11a}$$

$$\sigma\frac{\partial v}{\partial\tau} + \left(\frac{dV}{dr} + \frac{V}{r}\right)u = \left(DD_* + \frac{\partial^2}{\partial z^2}\right)v + Q(1+\epsilon\cos\tau)\frac{\partial b_\theta}{\partial z}, \tag{11b}$$

$$\sigma\frac{\partial w}{\partial\tau} = -\frac{\partial p}{\partial z} + \left(D_* D + \frac{\partial^2}{\partial z^2}\right)w + Q(1+\epsilon\cos\tau)\frac{\partial b_z}{\partial z}, \tag{11c}$$

$$\sigma\,\mathrm{Pm}\frac{\partial b_r}{\partial\tau} = (1+\epsilon\cos\tau)\frac{\partial u}{\partial z} + \left(DD_* + \frac{\partial^2}{\partial z^2}\right)b_r, \tag{11d}$$

$$\sigma\,\mathrm{Pm}\frac{\partial b_\theta}{\partial\tau} - \mathrm{Pm}\left(\frac{dV}{dr} - \frac{V}{r}\right)b_r = (1+\epsilon\cos\tau)\frac{\partial v}{\partial z} + \left(DD_* + \frac{\partial^2}{\partial z^2}\right)b_\theta, \tag{11e}$$

$$\sigma\,\mathrm{Pm}\frac{\partial b_z}{\partial\tau} = (1+\epsilon\cos\tau)\frac{\partial w}{\partial z} + \left(D_* D + \frac{\partial^2}{\partial z^2}\right)b_z, \tag{11f}$$

$$D_* u + \frac{\partial w}{\partial z} = 0, \tag{11g}$$

$$D_* b_r + \frac{\partial b_z}{\partial z} = 0, \tag{11h}$$

where

$$D = \frac{\partial}{\partial r}, \quad D_* = \frac{\partial}{\partial r} + \frac{1}{r},$$

and the Taylor and Chandrasekhar numbers are defined by

$$T = \frac{2\Omega_1^2 R_1^4}{\nu^2}, \quad Q = \frac{B_0^2 R_1^2}{\rho\mu\nu\lambda}. \tag{12}$$

Assuming that the boundaries are rigid and perfectly conducting, the appropriate boundary conditions, in terms of nondimensional variables, are [22]

$$u = v = w = b_r = D_* b_\theta = D b_z = 0 \quad \text{at } r = 1, 1/\eta. \tag{13}$$

We consider perturbations periodic in z, given by

$$u(r,z,\tau) = u(r,\tau)\cos az, \tag{14a}$$

$$v(r,z,\tau) = v(r,\tau)\cos az, \tag{14b}$$

$$w(r,z,\tau) = w(r,\tau)\sin az, \tag{14c}$$

$$p(r,z,\tau) = p(r,\tau)\cos az, \tag{14d}$$

$$b_r(r,z,\tau) = b_r(r,\tau)\sin az, \tag{14e}$$

$$b_\theta(r,z,\tau) = b_\theta(r,\tau)\sin az, \tag{14f}$$

$$b_z(r,z,\tau) = b_z(r,\tau)\cos az, \tag{14g}$$

Substituting in Eq. (11), we obtain

$$\left(M - \sigma\frac{\partial}{\partial\tau}\right)Mu + aQ(1+\epsilon\cos\tau)Mb_r = a^2 T\frac{V}{r}v, \tag{15a}$$

$$\left(M - \sigma\frac{\partial}{\partial\tau}\right)v + aQ(1+\epsilon\cos\tau)b_\theta = K_0 u, \tag{15b}$$

$$\left(M - \sigma\,\mathrm{Pm}\frac{\partial}{\partial\tau}\right)b_r = a(1+\epsilon\cos\tau)u, \tag{15c}$$

$$\left(M - \sigma\,\mathrm{Pm}\frac{\partial}{\partial\tau}\right)b_\theta + \mathrm{Pm}\frac{K_1}{r^2}b_r = a(1+\epsilon\cos\tau)v, \tag{15d}$$

$$u = Du = v = b_r = D_* b_\theta = 0 \quad \text{at } r = 1, 1/\eta, \tag{15e}$$

where

$$M = DD_* - a^2, \quad K_0 = \frac{dV}{dr} + \frac{V}{r} = -\frac{2\eta^2}{1-\eta^2},$$

$$K_1 = r^2\left(\frac{dV}{dr} - \frac{V}{r}\right) = -\frac{2}{1-\eta^2}.$$

We observe that the linear stability is governed by a system of equations with coefficients that are periodic functions of time. To obtain the stability boundaries we follow a procedure similar to that in [6,14]. On the stability boundaries the solution has to be periodic with the same time period as the coefficients in the system of equations. Therefore, we can assume the expansions

$$u(r,\tau) = u_s(r) + \sum_{n=1}^{\infty}\frac{1}{2}[u_n(r)e^{in\tau} + \tilde{u}_n(r)e^{-in\tau}], \tag{16a}$$

$$v(r,\tau) = v_s(r) + \sum_{n=1}^{\infty}\frac{1}{2}[v_n(r)e^{in\tau} + \tilde{v}_n(r)e^{-in\tau}], \tag{16b}$$

$$b_r(r,\tau) = b_{rs}(r) + \sum_{n=1}^{\infty}\frac{1}{2}[b_{rn}(r)e^{in\tau} + \tilde{b}_{rn}(r)e^{-in\tau}], \tag{16c}$$

$$b_\theta(r,\tau) = b_{\theta s}(r) + \sum_{n=1}^{\infty}\frac{1}{2}[b_{\theta n}(r)e^{in\tau} + \tilde{b}_{\theta n}(r)e^{-in\tau}], \tag{16d}$$

where the tilde is used to represent the complex conjugate. For $\epsilon \ll 1$, following Refs. [6,14] we assume

$$u_s = u_s^{(0)} + \epsilon^2 u_s^{(2)} + \cdots, \tag{17a}$$

$$v_s = v_s^{(0)} + \epsilon^2 v_s^{(2)} + \cdots, \tag{17b}$$

$$b_{rs} = b_{rs}^{(0)} + \epsilon^2 b_{rs}^{(2)} + \cdots, \tag{17c}$$

$$b_{\theta s} = b_{\theta s}^{(0)} + \epsilon^2 b_{\theta s}^{(2)} + \cdots, \tag{17d}$$

$$T_c = T_c^{(0)} + \epsilon^2 T_c^{(2)} + \cdots, \tag{17e}$$

$$u_1 = \epsilon u_1^{(1)} + \cdots, \tag{17f}$$

$$v_1 = \epsilon v_1^{(1)} + \cdots, \tag{17g}$$

$$b_{r1} = \epsilon b_{r1}^{(1)} + \cdots, \tag{17h}$$

$$b_{\theta 1} = \epsilon b_{\theta 1}^{(1)} + \cdots, \tag{17i}$$

where T_c is the critical Taylor number, i.e., the Taylor number at the point of marginal stability. Substituting from Eqs. (16) and (17) in Eq. (15) and separating to various orders in ϵ, we obtain to order ϵ^0

$$M^2 u_s^{(0)} - a^2 T_c^{(0)}\frac{V}{r}v_s^{(0)} + aQMb_{rs}^{(0)} = 0, \tag{18a}$$

$$K_0 u_s^{(0)} - Mv_s^{(0)} - aQb_{\theta s}^{(0)} = 0, \tag{18b}$$

$$au_s^{(0)} - Mb_{rs}^{(0)} = 0, \tag{18c}$$

$$av_s^{(0)} - \mathrm{Pm}\frac{K_1}{r^2}b_{rs}^{(0)} - Mb_{\theta s}^{(0)} = 0, \tag{18d}$$

$$u_s^{(0)} = Du_s^{(0)} = v_s^{(0)} = b_{rs}^{(0)} = D_* b_{\theta s}^{(0)} = 0 \quad \text{at } r = 1, 1/\eta. \tag{18e}$$

To order ϵ^1 we obtain

$$(M - i\sigma)Mu_1^{(1)} - a^2 T_c^{(0)} \frac{V}{r} v_1^{(1)} + aQMb_{r1}^{(1)} = -aQMb_{rs}^{(0)}, \tag{19a}$$

$$K_0 u_1^{(1)} - (M - i\sigma)v_1^{(1)} - aQb_{\theta 1}^{(1)} = aQb_{\theta s}^{(0)}, \tag{19b}$$

$$au_1^{(1)} - (M - i\sigma\,\mathrm{Pm})b_{r1}^{(1)} = -au_s^{(0)}, \tag{19c}$$

$$av_1^{(1)} - \mathrm{Pm}\frac{K_1}{r^2} b_{r1}^{(1)} - (M - i\sigma\,\mathrm{Pm})b_{\theta 1}^{(1)} = -av_s^{(0)}, \tag{19d}$$

$$u_1^{(1)} = Du_1^{(1)} = v_1^{(1)} = b_{r1}^{(1)} = D_* b_{\theta 1}^{(1)} = 0 \quad \text{at } r = 1, 1/\eta. \tag{19e}$$

Collecting the terms of order ϵ^2 that are independent of τ, we obtain

$$M^2 u_s^{(2)} - a^2 T_c^{(0)} \frac{V}{r} v_s^{(2)} + aQMb_{rs}^{(2)} = a^2 T_c^{(2)} \frac{V}{r} v_s^{(0)} - \frac{1}{4} aQM(b_{r1}^{(1)} + \tilde{b}_{r1}^{(1)}), \tag{20a}$$

$$K_0 u_s^{(2)} - Mv_s^{(2)} - aQb_{\theta s}^{(2)} = \frac{1}{4} aQ(b_{\theta 1}^{(1)} + \tilde{b}_{\theta 1}^{(1)}), \tag{20b}$$

$$au_s^{(2)} - Mb_{rs}^{(2)} = -\frac{1}{4} a(u_1^{(1)} + \tilde{u}_1^{(1)}), \tag{20c}$$

$$av_s^{(2)} - \mathrm{Pm}\frac{K_1}{r^2} b_{rs}^{(2)} - Mb_{\theta s}^{(2)} = -\frac{1}{4} a(v_1^{(1)} + \tilde{v}_1^{(1)}), \tag{20d}$$

$$u_s^{(2)} = Du_s^{(2)} = v_s^{(2)} = b_{rs}^{(2)} = D_* b_{\theta s}^{(2)} = 0 \quad \text{at } r = 1, 1/\eta. \tag{20e}$$

This inhomogeneous system has a unique solution provided the solvability condition is satisfied, which requires that the inhomogeneous term has to be orthogonal to the solution of the adjoint homogeneous problem. With the inner product defined by $(g,f) = \int_1^{1/\eta} g^T f r\, dr$, where f and g represent vectors of the form (u, v, b_r, b_θ), the adjoint problem is

$$M^2 \bar{u}_s^{(0)} + K_0 \bar{v}_s^{(0)} + a\bar{b}_{rs}^{(0)} = 0, \tag{21a}$$

$$-a^2 T_c^{(0)} \frac{V}{r} \bar{u}_s^{(0)} - M\bar{v}_s^{(0)} + a\bar{b}_{\theta s}^{(0)} = 0, \tag{21b}$$

$$aQM\bar{u}_s^{(0)} - M\bar{b}_{rs}^{(0)} - \mathrm{Pm}\frac{K_1}{r^2} \bar{b}_{\theta s}^{(0)} = 0, \tag{21c}$$

$$-aQ\bar{v}_s^{(0)} - M\bar{b}_{\theta s}^{(0)} = 0, \tag{21d}$$

$$\bar{u}_s^{(0)} = D\bar{u}_s^{(0)} = \bar{v}_s^{(0)} = \bar{b}_{rs}^{(0)} = D_* \bar{b}_{\theta s}^{(0)} = 0 \quad \text{at } r = 1, 1/\eta. \tag{21e}$$

The solvability condition leads to

$$T_c^{(2)} = \frac{1}{4a} \int_1^{1/\eta} [\bar{u}_s^{(0)} QM(b_{r1}^{(1)} + \tilde{b}_{r1}^{(1)}) - \bar{v}_s^{(0)} Q(b_{\theta 1}^{(1)} + \tilde{b}_{\theta 1}^{(1)}) + \bar{b}_{rs}^{(0)}(u_1^{(1)} + \tilde{u}_1^{(1)}) + \bar{b}_{\theta s}^{(0)}(v_1^{(1)} + \tilde{v}_1^{(1)})] r\, dr \Bigg/ \int_1^{1/\eta} \bar{u}_s^{(0)} \frac{V}{r} v_s^{(0)} r\, dr. \tag{22}$$

Substituting for $Mb_{r1}^{(1)}$ and $M\tilde{b}_{r1}^{(1)}$ from Eq. (19c) and its complex conjugate, we obtain

$$T_c^{(2)} = \frac{1}{4a} \int_1^{1/\eta} \{\bar{u}_s^{(0)} Q[i\sigma\,\mathrm{Pm}(b_{r1}^{(1)} - \tilde{b}_{r1}^{(1)}) + a(u_1^{(1)} + \tilde{u}_1^{(1)}) + 2au_s^{(0)}] - \bar{v}_s^{(0)} Q(b_{\theta 1}^{(1)} + \tilde{b}_{\theta 1}^{(1)}) + \bar{b}_{rs}^{(0)}(u_1^{(1)} + \tilde{u}_1^{(1)}) + \bar{b}_{\theta s}^{(0)}(v_1^{(1)} + \tilde{v}_1^{(1)})\} r\, dr \Bigg/ \int_1^{1/\eta} \bar{u}_s^{(0)} \frac{V}{r} v_s^{(0)} r\, dr. \tag{23}$$

III. RESULTS

As a first check we verify that our order-ϵ^0 calculations reproduce the results for the unmodulated system, given in Ref. [22]. Figure 1 shows $T_c^{(0)}$ versus Q for three different values of η, viz., 0.95, 0.85, and 0.75, assuming $\mathrm{Pm} = 10^{-5}$. Comparing, we observe that the three graphs have a similar nature but the values of $T_c^{(0)}$ and Q are different for different values of η. However, this apparent difference is because of the way in which we have defined T and Q. This becomes clear if we follow Chandrasekhar [22] and use $d = R_2 - R_1$ as the length scale, and define

$$T = -\frac{4C_1 \Omega_1 d^4}{\nu^2}, \quad Q = \frac{B_0^2 d^2}{\rho\mu\nu\lambda}, \tag{24}$$

where C_1 is the dimensional constant defined after Eq. (5). The three graphs in Fig. 1 are replotted, using these definitions of T and Q, in Fig. 2, and we observe that the three graphs almost coincide. This shows that the definitions in Eq. (24) are the appropriate nondimensional parameters for this problem. Two different definitions of T were used in Ref. [22] in the study of hydrodynamic cylindrical Couette flow. While a definition similar to Eq. (12) was used for a wide gap, for a narrow gap it was shown that the gap width should be chosen as the length scale and then the appropriate definition of the Taylor number is that given in Eq. (24). With this choice the equations governing linear stability and consequently the critical Taylor number become independent of η. Since we do not make a narrow-gap approximation we have used the expression in Eq. (12). A similar choice was made for the modulated hydrodynamic Couette flow with arbitrary gap widths in Ref. [14]. However, if we switch to

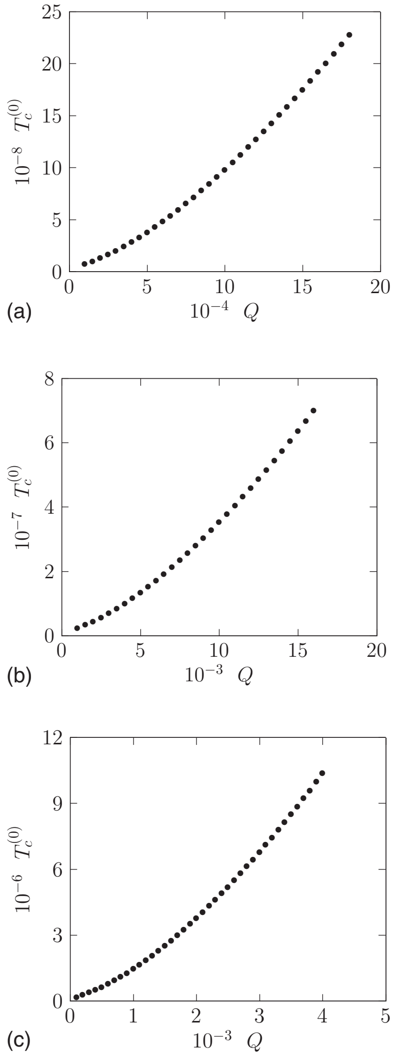

FIG. 1. $T_c^{(0)}$ as a function of Q for Pm=10^{-5} and η=(a) 0.95, (b) 0.85, and (c) 0.75. T and Q are as defined in Eq. (12).

the definitions in Eq. (24) we would expect our results to be independent of η provided the narrow gap approximation is valid. Our computed results show that even for η=0.75 the deviation from the narrow gap result is not very large. Further the results in Fig. 2 are in good agreement with the results in Ref. [22] and tend toward the narrow gap results as $\eta \rightarrow 1$.

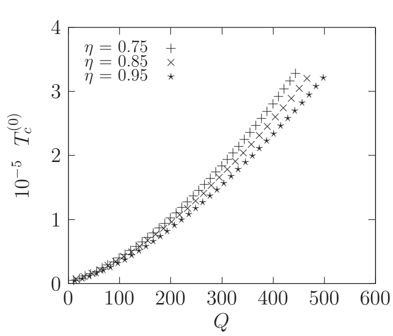

FIG. 2. $T_c^{(0)}$ as a function of Q for Pm=10^{-5} and η=0.95, 0.85, and 0.75. T and Q are as defined in Eq. (24).

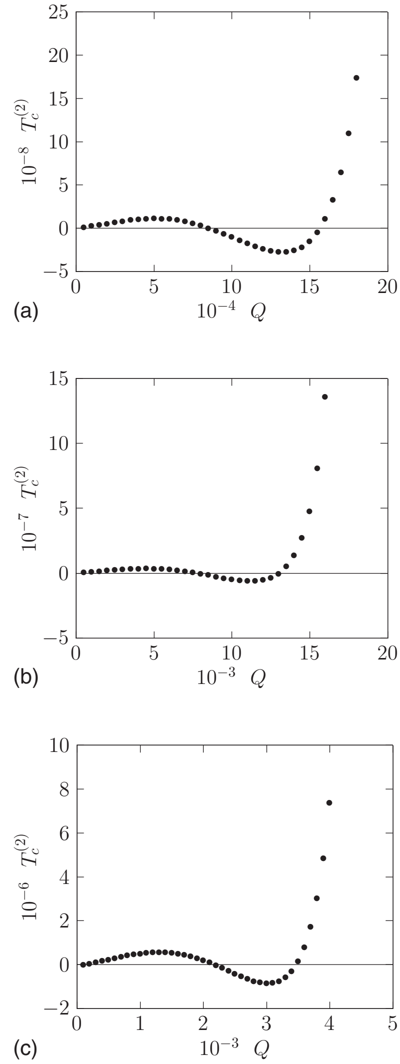

FIG. 3. $T_c^{(2)}$ as a function of Q for Pm=10^{-5}, σ=1 and η=(a) 0.95, (b) 0.85, and (c) 0.75. T and Q are as defined in Eq. (12).

We now consider the effect of modulation. Using Eq. (23) we have computed the shift in the critical Taylor number because of modulation of the imposed axial magnetic field, for various values of η, Pm, Q, and σ. The parameter values have been chosen so that they satisfy $\sigma\,\mathrm{Pm} \ll 1$, required for validity of the assumed form of the unperturbed magnetic field, given by Eq. (6). We first study the effect of varying the Chandrasekhar number Q. Figure 3 shows $T_c^{(2)}$ versus Q for three different values of η, viz., 0.95, 0.85, and 0.75. All three graphs have a similar nature. For low values of Q, modulation has a stabilizing effect, then for an intermediate range of Q it has a destabilizing effect, and for still higher values of Q it again has a stabilizing effect. However, comparing the three graphs, we observe that typical values of Q and $T_c^{(2)}$, say for the destabilizing range, are quite different for different values of η. Again, when the three graphs in Fig. 3 are replotted, using the definition of T and Q in Eq. (24), in Fig. 4 and we find that they almost coincide for low values of Q and are shifted by a small amount for larger values of Q. This again shows that the definitions in Eq. (24) are the appropriate nondimensional parameters for this prob-

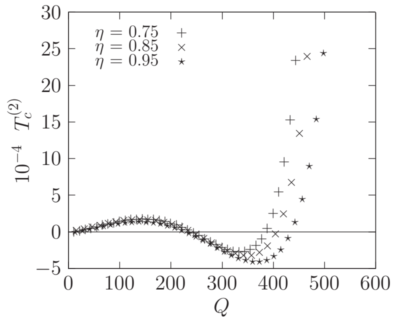

FIG. 4. $T_c^{(2)}$ as a function of Q for Pm=10^{-5}, σ=1, and η =0.95, 0.85, and 0.75. T and Q are as defined in Eq. (24).

lem. We next study how $T_c^{(2)}$ varies with η. This is shown in Fig. 5 for η in the range from 0.75 to 0.95. Figure 5(a) uses T and Q defined by Eq. (12) while Fig. 5(b) uses the definitions in Eq. (24). Once again we observe that using the definitions in Eq. (24) gives a more systematic trend. In this range of η, from Fig. 5(b) we observe that for Q=200 modulation is always stabilizing, for Q=300 it is always destabilizing, for Q=400 it is stabilizing for lower values of η but destabilizing for higher values, and for Q=450 it is again always stabilizing. Figures 3–5 are all for Pm=10^{-5} and σ =1. We have studied the effect of varying Pm in the range between 10^{-5} and 10^{-7} and σ for values $\sim$1. Typical results are shown in Table I, using the definitions in Eq. (12). We observe that the variation in $T_c^{(2)}$ is very small. So in this range $T_c^{(2)}$ may be considered to be practically independent of Pm and σ.

In computing $T_c^{(2)}$ we encountered a numerical problem. For solving Eqs. (18), (19), and (21) we used a second-order-

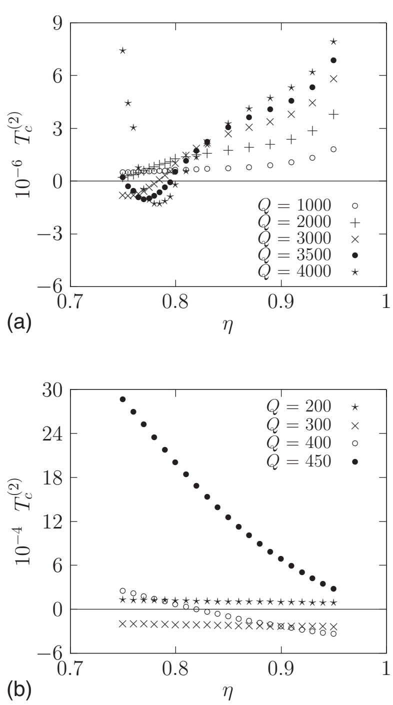

FIG. 5. $T_c^{(2)}$ as a function of η for different values of Q with Pm=10^{-5} and σ=1. T and Q defined by (a) Eq. (12) and (b) Eq. (24).

TABLE I. Variation of $T_c^{(2)}$ with Pm and σ.

Pm	σ	$T_c^{(2)}$
	η=0.95, $Q=5\times10^4$	
10^{-5}	1	109093655
10^{-5}	5	109093657
10^{-5}	10	109093663
10^{-6}	1	109076261
10^{-7}	1	109074522
	η=0.75, $Q=3\times10^3$	
10^{-5}	1	−810448
10^{-5}	5	−811078
10^{-5}	10	−813044
10^{-6}	1	−810033
10^{-7}	1	−809992

accurate finite-difference scheme with a set of uniformly spaced grid-points. We carried out the calculations in double precision in FORTRAN. On comparing results with different numbers of grid points we found that, while $T_c^{(0)}$ values showed the expected second-order convergence, the $T_c^{(2)}$ values did not. Results of a sample run are shown in Table II. Here again the definitions in Eq. (12) are used. A quantity that shows second-order convergence follows the equation $f_N=f+C/(N+1)^2+\cdots$, where f is the exact value, f_N is the value computed using N internal grid points, and C is a constant. With N internal grid points the inter-grid-point spacing scales as $1/(N+1)$. Consequently we must have

$$\frac{f_{N_3}-f_{N_2}}{f_{N_2}-f_{N_1}}=\frac{\dfrac{1}{(N_3+1)^2}-\dfrac{1}{(N_2+1)^2}}{\dfrac{1}{(N_2+1)^2}-\dfrac{1}{(N_1+1)^2}}$$

and, therefore, $(f_{99}-f_{89})/(f_{89}-f_{79})=0.715\,294$. We have $(T_{c,99}^{(0)}-T_{c,89}^{(0)})/(T_{c,89}^{(0)}-T_{c,79}^{(0)})=0.715\,422$ and, therefore, second-order convergence is confirmed. However, $T_c^{(2)}$ clearly does not obey this scaling and it is also seen that the $T_c^{(2)}$ values have not converged at all. On closer examination it was found that the trouble was with the order-ϵ^1 system,

TABLE II. $T_c^{(0)}$ and $T_c^{(2)}$ computed using double and quadruple precision for η=0.95, $Q=10^4$, Pm=10^{-5}, and σ=1.

N	$T_c^{(0)}$	$T_c^{(2)}$
	Double precision	
79	68092690	−77502255
89	68106145	239158575
99	68115771	−49522400
	Quadruple precision	
79	68092687	22262030
89	68106139	22264703
99	68115766	22266616

Eq. (19). Its solution did not show the expected proportionality to the zeroth-order driving terms. This occurred because the coefficient matrix of this system of equations was ill conditioned. The inverse of the condition number for 99 internal grid points is 9.0×10^{-20}. This value is such that computation done in double precision would not provide an accurate solution but computation in quadruple precision should. We redid the computation in quadruple precision and the results are given in Table II. We find $(T_{c,99}^{(0)}-T_{c,89}^{(0)})/(T_{c,89}^{(0)}-T_{c,79}^{(0)})=0.715\,656$ and $(T_{c,99}^{(2)}-T_{c,89}^{(2)})/(T_{c,89}^{(2)}-T_{c,79}^{(2)})=0.717\,978$. Thus both $T_c^{(0)}$ and $T_c^{(2)}$ now show second-order convergence. The small deviation from the expected scaling is due to higher-order corrections. Thus the numerical problem that we encountered because of the large condition number of the coefficient matrix of the order-ϵ^1 system has been solved by using quadruple precision. For the results reported in this paper we have used quadruple precision and 159 internal grid points, which was found to provide sufficient accuracy. With 159 internal grid points and the parameter values used in Table II, the inverse of the condition number for the coefficient matrix of the order-ϵ^1 system is 1.37×10^{-20} and, therefore, quadruple precision is expected to give accurate results.

One limitation of our analysis is the low-frequency approximation, $\sigma\,\mathrm{Pm}\ll 1$, which was made so that the equation for the unperturbed magnetic field, Eq. (3c), has a simple solution given by Eq. (6). This would not be valid if the modulation frequency were not small compared to (magnetic diffusion time)$^{-1}$. For modulation frequencies of the order of (magnetic diffusion time)$^{-1}$ or higher, Eq. (3c) would need to be solved without any approximation. This could form the subject of a future study.

Another limitation is that we make the assumption, common in theoretical studies, that the cylinders are infinitely long. In an actual experiment the cylinders would be of finite length and with a liquid as a working fluid the ends would have to be closed by end caps. Consequently, end-cap effects would be present. However, as a first step, we have assumed infinitely long cylinders, to allow us to study the effect of modulation of the magnetic field without getting mixed up with complications due to end-cap effects. But end-cap effects would undoubtedly be important, especially for comparison with experiments. Therefore, an extension of the present work to include end-cap effects can be taken up in a future study.

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